

Quantum fields in toroidal topology

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Abstract

The standard representation of c^* -algebra is used to describe fields in compactified space-time dimensions characterized by topologies of the type $\Gamma_D^d = (\mathbb{S}^1)^d \times \mathbb{M}^{D-d}$. The modular operator is generalized to introduce representations of isometry groups. The Poincaré symmetry is analyzed and then we construct the modular representation by using linear transformations in the field modes, similar to the Bogoliubov transformation. This provides a mechanism for compactification of the Minkowski space-time, that follows as a generalization of the Fourier-integral representation of the propagator at finite temperature. An important result is that the 2×2 representation of the real time formalism is not needed. The end result on calculating observables is described as a condensate in the ground state. We analyze initially the free Klein-Gordon and Dirac fields, and then formulate non-abelian gauge theories in Γ_D^d . Using the S-matrix, the decay of particles is calculated in order to show the effect of the compactification.

Key words: Quantum fields, Topology, Compactification, c^* - and Lie algebra

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1 Introduction

The thermal quantum field theory was proposed by Matsubara [1], based on an imaginary-time defined by a Wick rotation of the time-axis. The propagator is written as a Fourier series in the imaginary time, by using the frequencies: $\omega_n = 2\pi n/\beta$ ($\pi(2n+1)/\beta$) for bosons (fermions), corresponding to the period $\beta = T^{-1}$, with T being the temperature [2,3]. Using the notion of spectral function, as discussed by Kadanof and Baym [4], Dolan and Jackiw [5] managed to write the thermal propagator in a Fourier integral representation, thus restoring then time as a real quantity. In other words, the imaginary-time formalism is interpreted topologically. It is proved that the temperature is introduced in a quantum field theory by writing the original theory, formulated in the Minkowski space, \mathbb{M}^4 , in the compactified manifold $\Gamma_4^1 = \mathbb{S}^1 \times \mathbb{M}^3$, where the compactified dimension is the imaginary time. The circumference of \mathbb{S}^1 is β [6,7]. It is our objective here to generalize the argument advanced in Refs. [4] and [5] to include not only the time but also space coordinates, in such way that any set of dimensions of the manifold \mathbb{M}^D can be compactified, defining a theory in the topology $\Gamma_D^d = (\mathbb{S}^1)^d \times \mathbb{M}^{D-d}$, with $1 \leq d \leq D$. This establishes that the Fourier integral representation is sufficient to deal with the general question of compactification in the Γ_D^d topology at finite-temperature and real-time. To proceed, we have to set forth the general structure for such an approach. As a consequence, for the case of Γ_4^1 , that structure provides a simplified version of the real-time formalism for finite-temperature quantum fields.

It is well-known that there are two versions for a real-time finite-temperature quantum field theory. One was formulated by Schwinger [8,9,10] and Keldysh [11], and is based on using a path in the complex time plane [12]. The other is the thermofield dynamics (TFD) proposed by Takahashi and Umezawa [13]. In this case, the thermal theory is constructed on a Hilbert space and thermal effects are introduced by a Bogoliubov transformation [7,13,14,15]. In equilibrium, these two real-time formalisms are basically the same and the propagator, G^{ab} , $a, b = 1, 2$, is a 2×2 matrix [16]. However, the physical content is present only in the component G^{11} , which is, moreover, just the propagator in the Fourier integral representation as studied by Dolan and Jackiw. Such a result suggests that a real-time theory may be fully formulated in such a way that the propagator is written as a c -number (not as a matrix). This is an additional motivation for developing the central theme of this paper, insofar as this procedure can be extended to general cases with the topology Γ_D^d .

An approach describing systems in compactified spaces is derived as a generalization of both the Matsubara formalism, involving the Fourier series, [6,17,18,19] and TFD [20]. There are numerous applications of such a formalism, including the Casimir effect for the electromagnetic and fermion fields within a

box [20,21], the $\lambda\phi^4$ model describing the order parameter for the Ginsburg-Landau theory for superconductors [22], and the Gross-Neveu model as an effective approach for QCD [23,24]. The extension of this method to the Fourier integral representation is important to address many other problems in a topology Γ_D^d that are of interest in different areas, such as cosmology, condensed matter and particle physics [25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43]. In order to proceed with such a generalization, we rely on algebraic bases, using the modular representation of the c^* -algebra.

The c^* -algebra has played a central role in the development of functional analysis, and has attracted much attention due to its importance in non-commutative geometry [44,45,46,47]. It was first associated with the quantum field theory at finite temperature, through the imaginary time formalism [48]. Actually, the search for the algebraic structure of the Gibbs ensemble theory leads to the Tomita-Takesaki, or the standard representation of the c^* -algebra [48,49,50].

For the real-time formalism, the c^* -algebra approach was first analyzed by Ojima [51]. Later, the use of the Tomita-Takesaki Hilbert space, as the carrier space for representations of kinematical groups, was developed [52,53], in order to build thermal theories from symmetry: that is called the thermo group. A result emerging from this analysis is that the tilde-conjugation rules, the doubling in TFD, are identified with the modular conjugation of the standard representation. In addition, the Bogoliubov transformation, the other basic TFD ingredient, corresponds to a linear mapping involving the commutants of the von Neumann algebra. This TFD apparatus has been developed and applied in a wide range of problems [14,15,54]. In particular, the physical interpretation of the doubling of the operators has been identified [7].

The doubling of the propagator in real-time formalisms, however, is no longer necessary since we consider, as a starting point, the modular group introduced in a c^* -algebra. This is a result that we prove here, by using the modular representation of the Poincaré group. It is worth mentioning that, the modular conjugation is defined in order to respect the Lie algebra structure. This procedure provides a consistent way to define the modular conjugation for fermions; which is usually a non-simple task due to a lack of criterion [51]. Using a Bogoliubov transformation, the effect of compactification in Γ_D^d is introduced and a Fourier integral representation for the propagator is derived. Initially, an analysis is carried out for free boson and fermion fields. An extension of the formalism for abelian and non-abelian gauge-fields is introduced by functional methods. Exploring the canonical formalism, the S-matrix is developed and applied to calculate, as an example, decay rates by considering compactified spatial dimensions. It is important to emphasize that, the Feynman rules follow in parallel as in the Minkowski space-time theory and that the compactification corresponds to a process of condensation in the vacuum state.

The topology then leads naturally to the notion of quasi-particles.

The paper is organized in the following way. In Section 2, we present elements of the standard representation of the c^* -algebra, and use it to construct a representation theory of quantum fields in Γ_D^d topologies. In Section 3, the modular representations of Lie algebras are developed. In Section 4, physical aspects of the theory are discussed. In Section 5, we construct the path integral for determining fields in compactified space-time. In section 6, we elaborate the extension of the formalism to an $SU(3)$ gauge theory. In Section 7, the S -matrix is developed with application to the analysis of reaction rates. Concluding remarks are presented in Section 8.

2 C^* -algebra and compactified propagators

Let us initially present a résumé of some aspects of c^* -algebras in order to explore representations of Lie groups [52,53]. This sets the basis to built up the appropriate generalization of the finite-temperature formalism to also accommodate spatial compactification.

A c^* -algebra \mathcal{A} is a von Neumann algebra over the field of complex numbers \mathbb{C} with two different maps, an involutive mapping $*$: $\mathcal{A} \rightarrow \mathcal{A}$ and the *norm*, which is a mapping defined by $\|\cdot\| : \mathcal{A} \rightarrow \mathbb{R}_+$ [48,49,50]. Let $(\mathcal{H}_w, \pi_w(\mathcal{A}))$ be a faithful realization of \mathcal{A} , where \mathcal{H}_w is a Hilbert space and $\pi_w(\mathcal{A}) : \mathcal{H}_w \rightarrow \mathcal{H}_w$ is a $*$ -isomorphism of \mathcal{A} defined by linear operators in \mathcal{H}_w . Taking $|\xi_w\rangle \in \mathcal{H}_w$ to be normalized, it follows that $\langle \xi_w | \pi_w(A) | \xi_w \rangle$, for every $A \in \mathcal{A}$, defines a state over \mathcal{A} denoted by $w_\xi(A) = \langle \xi_w | \pi_w(A) | \xi_w \rangle$. As was demonstrated by Gel'fand, Naimark and Segal (GNS), the inverse is also true; i.e. every state ω of a c^* -algebra \mathcal{A} admits a vector representation $|\xi_w\rangle \in \mathcal{H}_w$ such that $w(A) \equiv \langle \xi_w | \pi_w(A) | \xi_w \rangle$. This realization is called the GNS construction [48,49,50], which is valid if the dual coincides with the pre-dual.

The Tomita-Takesaki (standard) representation is a class of representations of c^* -algebras introduced as follows. Consider $\sigma : \mathcal{H}_w \rightarrow \mathcal{H}_w$ to be a (modular) conjugation in \mathcal{H}_w , that is, σ is an anti-linear isometry such that $\sigma^2 = 1$. The set $(\mathcal{H}_w, \pi_w(\mathcal{A}))$ is a Tomita-Takesaki representation of \mathcal{A} , if $\sigma\pi_w(\mathcal{A})\sigma = \tilde{\pi}_w(\mathcal{A})$ defines a $*$ -anti-isomorphism on the linear operators. It follows that $(\mathcal{H}_w, \tilde{\pi}_w(\mathcal{A}))$ is a faithful anti-realization of \mathcal{A} . It is to be noted that $\tilde{\pi}_w(\mathcal{A})$ is the commutant of $\pi_w(\mathcal{A})$; i.e. $[\pi_w(\mathcal{A}), \tilde{\pi}_w(\mathcal{A})] = 0$. In this representation, the state vectors are invariant under σ ; that is, $\sigma |\xi_w\rangle = |\xi_w\rangle$. As long as there is no confusion, elements of the set $\pi_w(\mathcal{A})$ will be denoted by A and those of $\tilde{\pi}_w(\mathcal{A})$ by \tilde{A} .

With this notation, the tilde and non-tilde operators, defined above by the σ

modular conjugation, have the properties,

$$\begin{aligned}
(A_i A_j)^\sim &= \tilde{A}_i \tilde{A}_j, \\
(c A_i + A_j)^\sim &= c^* \tilde{A}_i + \tilde{A}_j, \\
(A_i^\dagger)^\sim &= (\tilde{A}_i)^\dagger, \\
\tilde{\tilde{A}}_i &= A_i, \\
[A_i, \tilde{A}_j] &= 0, \\
|\xi_w\rangle^\sim &= |\xi_w\rangle \text{ and } \langle \xi_w|^\sim = \langle \xi_w|.
\end{aligned}$$

These tilde-conjugation rules are derived in TFD in association with properties of physical (usually non-interacting) systems and are called the tilde conjugation rules [14].

An interesting aspect of this construction is that properties of $*$ -automorphisms in \mathcal{A} can be defined through a unitary operator, say $\Delta(\tau)$, invariant under the modular conjugation, i.e. $[\Delta(\tau), \sigma] = 0$. Then writing $\Delta(\tau)$ as $\Delta(\tau) = \exp(i\tau \hat{A})$, where \hat{A} is the generator of symmetry, we have $\sigma \hat{A} \sigma = -\hat{A}$. Therefore, the generator \hat{A} is an odd polynomial function of $A - \tilde{A}$, i.e.

$$\hat{A} = f(A - \tilde{A}) = \sum_{n=0}^{\infty} c_n (A - \tilde{A})^{2n+1}, \quad (1)$$

where the coefficients $c_n \in \mathbb{R}$.

Consider the simple case where $c_0 = 1$, $c_n = 0, \forall n \neq 0$, i.e. $\hat{A} = A - \tilde{A}$. Taking A to be the Hamiltonian, H , the time-translation generator is given by \hat{H} . The parameter τ is related to a Wick rotation such that $\tau \rightarrow i\beta$; resulting in $\Delta(\beta) = e^{-\beta \hat{H}}$, where $\beta = T^{-1}$, T being the temperature. This is the so-called modular operator in c^* -algebras. As a consequence, a realization for $w(A)$ as a Gibbs ensemble average is introduced [48,49,50],

$$w_A^\beta = \frac{\text{Tr}(e^{-\beta H} A)}{\text{Tr} e^{-\beta H}}. \quad (2)$$

We now proceed with the generalization of this construction for finite temperature, corresponding to the compactification of the imaginary time, to accommodate also spatial compactification. To do so, we replace H by the generator of space-time translations, P^μ , in a d -dimensional subspace of a D -dimensional Minkowski space-time, \mathbb{M}^D , with $d \leq D$. Then we generalize Eq. (2) to the form

$$w_A^\alpha = \frac{\text{Tr}(e^{-\alpha_\mu P^\mu} A)}{\text{Tr} e^{-\alpha_\mu P^\mu}}, \quad (3)$$

where α_μ are the group parameters. This leads to the following statement:

- Proposition 1: For $A(x)$ in the c^* -algebra \mathcal{A} , there is a function $w_A^\alpha(x)$, x in \mathbb{M}^D , defined by Eq. (3) such that

$$w_A^\alpha(x) = w_A^\alpha(x + i\alpha), \quad (4)$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{d-1}, 0, \dots, 0)$. This implies that $w_A^\alpha(x)$ is periodic, in the d -dimensional subspace, with α_0 being the period in the imaginary time, i.e. $\alpha_0 = \beta$, and $\alpha_j = iL_j$, $j = 1, \dots, d-1$, are identified with the periodicity in spatial coordinates.

It is important to be noted that $w_A^\alpha(x)$ preserves the isometry, since it is defined by elements of the isometry group. Therefore, the theory is defined in the topology $\Gamma_D^d = (\mathbb{S}^1)^d \times \mathbb{M}^{D-d}$. For the particular case of $d = 1$, taking $\alpha_0 = \beta$, we have to identify Eq. (4) as the KMS (Kubo-Martin-Schwinger) condition [7]. Then by using the GNS construction, a quantum theory in thermal equilibrium is equivalent to taking this theory in a Γ_D^1 topology in the imaginary-time axis, where the circumference of \mathbb{S}^1 is β , as it is well known. The generalization of this result for space coordinates is given by Eq. (4); it corresponds to the generalized KMS condition for field theories in toroidal spaces $(\mathbb{S}^1)^d \times \mathbb{M}^{D-d}$. On the other hand the average given in Eq. (3) can also be written as

$$w_A^\alpha(x) \equiv \langle \xi_w^\alpha | A(x) | \xi_w^\alpha \rangle. \quad (5)$$

In the next section, we turn our attention to constructing the state $|\xi_w^\alpha\rangle$ explicitly.

Let us consider, as an example, the free propagator for the Klein-Gordon-field. In this case the Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{m^2}{2} \phi(x)^2,$$

and we take

$$A(x, x') = T[\phi(x)\phi(x')],$$

where T is the time-ordering operator. The propagator for the compactified field is $G_0(x - x'; \alpha) \equiv w_A^\alpha(x, x')$, that is

$$G_0(x - x'; \alpha) = \frac{\text{Tr}(e^{-\alpha_\mu P^\mu} T\phi(x)\phi(x'))}{\text{Tr}e^{-\alpha_\mu P^\mu}}. \quad (6)$$

The generalized KMS condition, Eq. (4), then allows us to write the series-integral representation, corresponding to a modified Matsubara prescription, i.e.

$$G_0(x - x'; \alpha) = \frac{1}{i^d \alpha_0 \cdots \alpha_{d-1}} \sum_{n_0 \cdots n_{d-1}} \int \frac{d^{D-d} \mathbf{k}}{(2\pi)^{D-d}} \frac{e^{-ik_\alpha \cdot (x-x')}}{k_\alpha^2 - m^2 + i\varepsilon}, \quad (7)$$

where $k_\alpha = (k_{n_0}^0, k_{n_1}^1, \dots, k_{n_{d-1}}^{d-1}, k^d \dots, k^{D-1})$, with

$$k_{n_j}^j = \frac{2\pi n_j}{\alpha_j}, \quad 0 \leq j \leq d-1,$$

$n_j \in \mathbb{Z}$ and $d^{D-d}\mathbf{k} = dk^d dk^{d+1} \dots dk^{D-1}$. In what follows, we employ the name Matsubara representation (or prescription) referring to both time and space compactification. The Green function $G_0(x - x'; \alpha)$ is a solution of the Klein-Gordon equation since w_A^α , by definition, respects the isometry. This means that $G_0(x - x'; \alpha)$ is the Green function of a boson field defined locally in the Minkowski space-time. Globally this theory is such that $G_0(x - x'; \alpha)$ has to satisfy periodic boundary conditions. These facts assure us that $G_0(x - x'; \alpha)$ is the Green function of a field theory defined in a hyper-torus, $\Gamma_D^d = (\mathbb{S}^1)^d \times \mathbb{M}^{D-d}$, with $1 \leq d \leq D$, where the circumference of the j -th \mathbb{S}^1 is specified by α_j . Then we can proceed to study representations in terms of the spectral function, as derived for the case of temperature by Dolan and Jackiw [5]. We first consider one-compactified dimension as an example, following the detailed calculation presented in the Appendix.

Take the topology Γ_D^1 where the imaginary time-axis is compactified. In this case, we denote, $\alpha = (\beta, 0, \dots, 0) = \beta \hat{n}_0$, $\hat{n}_0 = (1, 0, \dots, 0)$, with $T = \beta^{-1}$ being the temperature, such that the Green function is given by

$$G_0(x - y; \beta) = \frac{1}{i\beta} \sum_{l_0} \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{e^{-ik_{l_0} \cdot (x-y)}}{(k_{l_0})^2 - m^2 + i\varepsilon},$$

where $k_{l_0} = (k_{l_0}^0, k^1, \dots, k^{D-1})$, with $k_{l_0}^0 = 2\pi l_0 / \beta$, being the Matsubara frequency. Then this propagator is mapped, by an analytical continuation, into a Fourier integral representation given by

$$G_0(x - y; \beta) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-y)} G_0(k; \beta), \quad (8)$$

where

$$G_0(k; \beta) = G_0(k) + f_\beta(k^0)[G_0(k) - G_0^*(k)]$$

and

$$f_\beta(k^0) = \sum_{l_0=1}^{\infty} e^{-\beta \omega_k l_0} = \frac{1}{e^{\beta \omega_k} - 1} \equiv n(k^0; \beta),$$

which is the boson distribution function at temperature T with $\omega_k = k^0$. Then we have

$$G_0(k; \beta) = \frac{-1}{k^2 - m^2 + i\varepsilon} + n(k^0; \beta) 2\pi i \delta(k^2 - m^2),$$

In the case of compactification of the coordinate x^1 , for the topology Γ_D^1 , we take $\alpha = (0, iL_1, 0, \dots, 0) = iL_1 \hat{n}_1$, with $\hat{n}_1 = (0, 1, 0, \dots, 0)$. The factor i in the parameter α_j corresponding to the compactification of a space coordinate

makes explicit that we are working with the Minkowski metric; the period in the x^1 direction is real and equal to L_1 . The propagator has the Matsubara representation

$$G_0(x - y; L_1) = \frac{1}{L_1} \sum_{l_1} \int \frac{d^{D-1} \mathbf{k}}{(2\pi)^{D-1}} \frac{e^{-ik_{l_1} \cdot (x-y)}}{(k_{l_1})^2 - m^2 + i\varepsilon},$$

where $k_{l_1} = (k^0, k_{l_1}^1, k^2, \dots, k^{D-1})$, with $k_{l_1}^1 = 2\pi l_1/L_1$. The Fourier-integral representation can be derived along the same way as in the case of temperature [7], leading to

$$G_0(x - y; L_1) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-y)} G_0(k; L_1),$$

where

$$G_0(k; L_1) = \frac{-1}{k^2 - m^2 + i\varepsilon} + f_{L_1}(k^1) 2\pi i \delta(k^2 - m^2),$$

with

$$f_{L_1}(k^1) = \sum_{l_1=1}^{\infty} e^{-iL_1 k^1 l_1}.$$

As another example, we consider the topology Γ_D^2 , accounting for a double compactification, one being the imaginary time and the other the x^1 direction. In this case $\alpha = (\beta, iL_1, 0, \dots, 0) = \beta \hat{n}_0 + iL_1 \hat{n}_1$. The Matsubara representation is

$$G_0(x - y; \beta, L_1) = \frac{1}{i\beta L_1} \sum_{l_0, l_1} \int \frac{d^{D-2} \mathbf{k}}{(2\pi)^{D-2}} \frac{e^{-ik_{l_0 l_1} \cdot (x-x')}}{(k_{l_0 l_1})^2 - m^2 + i\varepsilon},$$

where $k_{l_0 l_1} = (k_{l_0}^0, k_{l_1}^1, k^2, \dots, k^{D-1})$, with $k_{l_0}^0 = 2\pi l_0/\beta$ and $k_{l_1}^1 = 2\pi l_1/L_1$. The corresponding Fourier-integral representations is

$$G_0(x - y; \beta, L_1) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-y)} G_0(k; \beta, L_1),$$

where

$$G_0(k; \beta, L_1) = \frac{-1}{k^2 - m^2 + i\varepsilon} + f_{\beta L_1}(k^0, k^1) 2\pi i \delta(k^2 - m^2),$$

with

$$f_{\beta L_1}(k^0, k^1) = f_{\beta}(k^0) + f_{L_1}(k^1) + 2f_{\beta}(k^0)f_{L_1}(k^1).$$

In the Appendix we demonstrate this result, as well as the generalization for d compactified dimensions. In any case, the general structure of the propagator is given by

$$G_0(x - y; \alpha) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-y)} G_0(k; \alpha), \quad (9)$$

where

$$G_0(k; \alpha) = G_0(k) + f_{\alpha}(k_{\alpha})[G_0(k) - G_0^*(k)], \quad (10)$$

This is one of the main result in this paper. The next step is to consider interacting systems in a topology Γ_D^d . Before that, we have to finish the GNS construction for this case.

3 *-Lie algebras and field theory

Our main goal in this section is to demonstrate the existence of the states $|\xi_w^\alpha\rangle$, introduced in Eq. (5), as a second part of the GNS construction. For that, we study first general elements of representations for Lie algebras, by using the modular representations of a c^* -algebra. Applying the results for the Poincaré group, we analyze representations describing free bosons and fermions compactified in space-time.

3.1 Modular representation and Lie groups

Consider $\ell = \{a_i, i = 1, 2, 3, \dots\}$ a Lie algebra over the (real) field \mathbb{R} , of a Lie group \mathcal{G} , characterized by the algebraic relations $(a_i, a_j) = C_{ijk}a_k$, where $C_{ijk} \in \mathbb{R}$ are the structure constants and $(,)$ is the Lie product (we are assuming the convention of summation over repeated indices). Using the modular conjugation, $*$ -representations for ℓ , denoted by $*\ell$, are constructed. Let us take $\pi(\ell)$, a representation for ℓ as a von Neumann algebra, and $\tilde{\pi}(\ell)$ as the representation for the correspondent commutant. Each element in ℓ is denoted by $\pi(a_i) = A_i$ and $\tilde{\pi}(a_i) = \tilde{A}_i$; thus we have [55],

$$[\tilde{A}_i, \tilde{A}_j] = -iC_{ijk}\tilde{A}_k, \quad (11)$$

$$[A_i, A_j] = iC_{ijk}A_k, \quad (12)$$

$$[\tilde{A}_i, A_j] = 0. \quad (13)$$

This provides a reducible representation for ℓ without an apparent physical or mathematical outcome of interest. However, a careful analysis brings out facts that are important, at least, in physics. The modular generators of symmetry are given by $\hat{A} = A - \tilde{A}$. Then we have from Eqs. (11)-(13) that the $*\ell$ algebra is given by

$$[\hat{A}_i, \hat{A}_j] = iC_{ijk}\hat{A}_k, \quad (14)$$

$$[\hat{A}_i, A_j] = iC_{ijk}A_k, \quad (15)$$

$$[A_i, A_j] = iC_{ijk}A_k. \quad (16)$$

This is just the semidirect product of the faithful representation $\pi(a_i) = A_i$ and the other faithful representation $\tilde{\pi}(A_i) = \tilde{A}_i$, with $\pi(a_i)$ providing elements of the invariant subalgebra. This is the proof of the following statement:

- Proposition 2. Consider the Tomita-Takesaki representation, where the von Neumann algebra is a Lie algebra, ℓ . Then the modular representation for ℓ is given by Eqs. (14)-(16), the $^*\ell$ -algebra, where the invariant subalgebra describes properties of observables of the theory, that are transformed under the symmetry defined by the generators of modular transformations.

Another aspect to be explored is a set of linear mappings $U(\xi) : \pi_w(\mathcal{A}) \times \tilde{\pi}_w(\mathcal{A}) \rightarrow \pi_w(\mathcal{A}) \times \tilde{\pi}_w(\mathcal{A})$ with the characteristics of a Bogoliubov transformation, i.e. $U(\xi)$ is canonical, in the sense of keeping the algebraic relations, and unitary but only for a finite dimensional basis. Then we have a group with elements $U(\xi)$ specified by the parameters ξ . This is due to the two commutant sets in the von Neumann algebra. The characteristic of $U(\xi)$ as a linear mapping is guaranteed by the canonical invariance of $^*\ell$. In terms of generators of symmetry and tilde operators we obtain,

$$\begin{aligned} A(\xi) &= U(\xi)AU(\xi)^{-1}, \\ \tilde{A}(\xi) &= U(\xi)\tilde{A}U(\xi)^{-1}, \end{aligned}$$

such that

$$\begin{aligned} [\tilde{A}(\xi)_i, \tilde{A}(\xi)_j] &= -iC_{ijk}\tilde{A}(\xi)_k, \\ [A(\xi)_i, A(\xi)_j] &= iC_{ijk}A(\xi)_k, \\ [\tilde{A}(\xi)_i, A(\xi)_j] &= 0. \end{aligned}$$

The goal here is to use $U(\xi)$ to construct explicitly the states $w_A^\xi(x)$ introduced in Eq. (4), describing fields in a Γ_D^d topology. For the Poincaré algebra, for instance, we have the * -Poincaré Lie algebra ($^*\mathfrak{p}$) given by

$$[M_{\mu\nu}, P_\sigma] = i(g_{\nu\sigma}P_\mu - g_{\sigma\mu}P_\nu), \quad (17)$$

$$[P_\mu, P_\nu] = 0, \quad (18)$$

$$[M_{\mu\nu}, M_{\sigma\rho}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}), \quad (19)$$

$$[\tilde{M}_{\mu\nu}, \tilde{P}_\sigma] = -i(g_{\nu\sigma}\tilde{P}_\mu - g_{\sigma\mu}\tilde{P}_\nu), \quad (20)$$

$$[\tilde{P}_\mu, \tilde{P}_\nu] = 0, \quad (21)$$

$$[\tilde{M}_{\mu\nu}, \tilde{M}_{\sigma\rho}] = i(g_{\mu\rho}\tilde{M}_{\nu\sigma} - g_{\nu\rho}\tilde{M}_{\mu\sigma} + g_{\mu\sigma}\tilde{M}_{\rho\nu} - g_{\nu\sigma}\tilde{M}_{\rho\mu}), \quad (22)$$

where $\tilde{M}_{\mu\nu} = \tilde{M}_{\mu\nu}(\xi)$, $M_{\mu\nu} = M_{\mu\nu}(\xi)$, $\tilde{P}_\mu = \tilde{P}_\mu(\xi)$ and $P_\mu = P_\mu(\xi)$. All other commutation relations are zero. For this algebra, we obtain representations for generators of symmetry. Generators of the Poincaré symmetry are given by $\tilde{\hat{M}}_{\mu\nu} = \tilde{M}_{\mu\nu} - \tilde{M}_{\mu\nu}$ and $\tilde{\hat{P}}_\mu = \tilde{P}_\mu - \tilde{P}_\mu$ and satisfy the commutation relations similar to those given by Eqs. (14)-(16). The representations are constructed

by using a set of Casimir invariants, i.e.

$$w^2 = w_\mu w^\mu, \quad (23)$$

$$P^2 = P_\mu P^\mu, \quad (24)$$

$$\widetilde{w}^2 = \widetilde{w}_\mu \widetilde{w}^\mu, \quad (25)$$

$$\widetilde{P}^2 = \widetilde{P}_\mu \widetilde{P}^\mu \quad (26)$$

where $w_\mu = \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}M^{\nu\sigma}P^\rho$ is the Pauli-Lubanski vector.

3.2 Boson fields

Let us consider a free quantum field describing bosons. The modular conjugation rules can be applied to any relation among the dynamical variables, in particular to the equation of motion in the Heisenberg picture. The set of doubled equations are then derived by writing the hat-Hamiltonian, the generator of time translation, as $\widehat{H} = H - \widetilde{H}$. However, due to the GNS construction, we need to consider only the evolution of the Lagrangian for non-tilde operators, evolving in space-time by the generators of ${}^*\mathfrak{p}$. In this case the time evolution generator is \widehat{H} . Then we have the Lagrangian densities $\mathcal{L}(x)$ and $\mathcal{L}(x; \xi)$ given, respectively, by

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2, \quad (27)$$

$$\mathcal{L}(x; \xi) = \frac{1}{2}\partial_\mu\phi(x; \xi)\partial^\mu\phi(x; \xi) - \frac{m^2}{2}\phi(x; \xi)^2, \quad (28)$$

where the field $\phi(x; \xi)$ is defined by

$$\phi(x; \xi) = U(\xi)\phi(x)U^{-1}(\xi).$$

The mapping $U(\xi)$ is taken as a Bogoliubov transformation and is defined, as usual, by a two-mode squeezed operator. For fields expanded in terms of modes, we define

$$\begin{aligned} U(\xi) &= \exp \left\{ \sum_k \theta(k_\xi; \xi) [a^\dagger(k)\widetilde{a}^\dagger(k) - a(k)\widetilde{a}(k)] \right\} \\ &= \prod_k U(k; \xi), \end{aligned} \quad (29)$$

where

$$U(k_\xi; \xi) = \exp\{\theta(k_\xi; \xi)[a^\dagger(k)\widetilde{a}^\dagger(k) - a(k)\widetilde{a}(k)]\},$$

with $\theta(k_\xi; \xi)$ being a function of the momentum, k_ξ , and of the parameters ξ , both to be specified. The label k in the sum and in the product of the

equations above is to be taken in the continuum limit, for each mode. Then we have

$$\phi(x; \xi) = \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{1}{2k_0} [a(k; \xi)e^{-ikx} + a^\dagger(k; \xi)e^{ikx}]. \quad (30)$$

To obtain this expression, we have used the non-zero commutation relations

$$[a(k; \xi), a^\dagger(k'; \xi)] = (2\pi)^3 2k_0 \delta(\mathbf{k} - \mathbf{k}'), \quad (31)$$

with

$$\begin{aligned} a(k; \xi) &= U(k_\xi; \xi) a(k) U^{-1}(k_\xi; \xi) \\ &= u(k_\xi; \xi) a(k) - v(k_\xi; \xi) \tilde{a}^\dagger(k), \end{aligned}$$

where $u(k_\xi; \xi)$ and $v(k_\xi; \xi)$ are given in terms of $\theta(k_\xi; \xi)$ by

$$\begin{aligned} u(k_\xi; \xi) &= \cosh \theta(k_\xi; \xi), \\ v(k_\xi; \xi) &= \sinh \theta(k_\xi; \xi). \end{aligned}$$

The inverse is

$$a(k) = u(k_\xi; \xi) a(k; \xi) + v(k_\xi; \xi) \tilde{a}^\dagger(k; \xi),$$

such that the other operators $a^\dagger(k)$, $\tilde{a}(k)$ and $\tilde{a}^\dagger(k)$ are obtained by applying the hermitian conjugation or the tilde conjugation, or both.

It is worth noting that the transformation $U(\xi)$ can be mapped into a 2×2 representation of the Bogoliubov transformation, i.e.

$$B(k_\xi; \xi) = \begin{pmatrix} u(k_\xi; \xi) & -v(k_\xi; \xi) \\ -v(k_\xi; \xi) & u(k_\xi; \xi) \end{pmatrix}, \quad (32)$$

with $u^2(k_\xi; \xi) - v^2(k_\xi; \xi) = 1$, acting on the pair of commutant operators as

$$\begin{pmatrix} a(k; \xi) \\ \tilde{a}^\dagger(k; \xi) \end{pmatrix} = B(k_\xi; \xi) \begin{pmatrix} a(k) \\ \tilde{a}^\dagger(k) \end{pmatrix}.$$

A Bogoliubov transformation of this type gives rise to a compact and elegant 2×2 representation of the propagator in the real-time formalism. However, to derive and use a quantum field theory in a topology Γ_D^d following the GNS construction, we observe that this matrix representation for the propagator is indeed not necessary. This aspect is useful for applications, in particular to represent an ease in the calculations of physical processes.

The Hilbert space is constructed from the ξ -state, $|0(\xi)\rangle = U(\xi)|0, \tilde{0}\rangle$, where $|0, \tilde{0}\rangle = \bigotimes_k |0, \tilde{0}\rangle_k$ and $|0, \tilde{0}\rangle_k$ is the vacuum for the mode k . Then we have: $a(k; \xi)|0(\xi)\rangle = \tilde{a}(k; \xi)|0(\xi)\rangle = 0$ and $\langle 0(\xi)|0(\xi)\rangle = 1$. This shows that $|0(\xi)\rangle$ is

a vacuum for ξ -operators $a(k; \xi)$. However, it is a condensate for the operators a and a^\dagger . An arbitrary basis vector is given in the form

$$|\psi(\xi); \{m\}; \{k\}\rangle = [a^\dagger(k_1; \xi)]^{m_1} \cdots [a^\dagger(k_M; \xi)]^{m_M} \times [\tilde{a}^\dagger(k_1; \xi)]^{n_1} \cdots [\tilde{a}^\dagger(k_N; \xi)]^{n_N} |0(\xi)\rangle, \quad (33)$$

where $n_i, m_j = 0, 1, 2, \dots$, with N and M being indices for an arbitrary mode.

Consider only one field-mode, for simplicity. Then we write $|0(\xi)\rangle$ in terms of $u(\xi)$ and $v(\xi)$ as

$$\begin{aligned} |0(\xi)\rangle &= \frac{1}{u(\xi)} \exp\left[\frac{v(\xi)}{u(\xi)} a^\dagger \tilde{a}^\dagger\right] |0, \tilde{0}\rangle \\ &= \frac{1}{u(\xi)} \sum_n \left(\frac{v(\xi)}{u(\xi)}\right)^n |n, \tilde{n}\rangle. \end{aligned} \quad (34)$$

This provides an explicit example of states in the GNS construction for a quantum field theory in a topology Γ_D^d . Since the state $|0(\xi)\rangle$ is a trace-like state, this leads to the state $w_A^\alpha(x)$. At this point, the physical meaning of an arbitrary ξ -state given in Eq. (33) is not established. This aspect is discussed by considering the Green function defined by

$$G_0(x - y; \xi) = -i \langle 0(\xi) | T[\phi(x) \phi(y)] | 0(\xi) \rangle.$$

We demand then that $G_0(x - y; \xi) \equiv G_0(x - y; \alpha)$, where $G_0(x - y; \alpha)$ is given in Eq. (6). Using $U(\xi)$ in Eq. (29), we find that the ξ -Green function is written as

$$G_0(x - y; \xi) = -i \langle \tilde{0}, 0 | T[\phi(x; \xi) \phi(y; \xi)] | 0, \tilde{0} \rangle.$$

Then, using the field expansion (30) and the commutation relation (31), we obtain

$$G_0(x - y; \xi) = \int \frac{d^D k}{(2\pi)^D} e^{-ik(x-y)} G_0(k; \xi), \quad (35)$$

where

$$G_0(k; \xi) = G_0(k) + v^2(k_\xi; \xi) [G_0(k) - G_0^*(k)].$$

This propagator is formally identical to $G_0(x - y; \alpha)$ written in the integral representation given by Eqs. (9) and (10). Then the analysis in terms of representation of Lie-groups and the Bogoliubov transformation leads to the integral representation by performing the mapping $v^2(k_\xi; \xi) \rightarrow f_\alpha(k_\alpha)$. Notice that this is possible, since $v^2(k_\xi; \xi)$ has not been fully specified up to this point. Considering the specific case of compactification in time, in order to describe temperature only, the real quantity $v^2(k_\xi; \xi)$ is mapped in the real quantity $f_\beta(w) \equiv n(\beta)$. Including space compactification, $f_\alpha(k_\alpha)$ is a complex function, according to Appendix. In such a case, we can consider $f_\alpha(k_\alpha)$ as the analytical continuation of the real function $v^2(k_\xi; \xi)$; a procedure that is possible,

since, $v^2(k_\xi; \xi)$ is arbitrary. Therefore, for space compactification, we can also perform the mapping $v^2(k_\xi; \xi) \rightarrow f_\alpha(k_\alpha)$ in $G_0(k; \xi)$, in order to recover the propagator shown in Eqs. (9) and (10). From now on, we denote the vector $|\xi_w^\alpha\rangle$ by $|\alpha\rangle$ and the function $f_\alpha(k_\alpha)$ by $v^2(k_\alpha; \alpha)$.

3.3 Fermion field

A similar mathematical structure is introduced for the compactification of fermion fields. With the average given in Eq. (3), $w_A^\alpha(x) \equiv \langle \alpha | A(x) | \alpha \rangle$, $A(x)$ is defined in terms of fermion operators. We have first to construct the state $|\alpha\rangle$ explicitly.

The Lagrangian density for the free Dirac field is

$$\mathcal{L}(x) = \frac{1}{2} \bar{\psi}(x) \left[\gamma \cdot i \overleftrightarrow{\partial} - m \right] \psi(x) \quad (36)$$

and for the α -field we have

$$\mathcal{L}(x; \alpha) = \frac{1}{2} \bar{\psi}(x; \alpha) \left[\gamma \cdot i \overleftrightarrow{\partial} - m \right] \psi(x; \alpha). \quad (37)$$

The field $\psi(x; \alpha)$ is expanded as

$$\psi(x; \alpha) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{m}{k_0} \sum_{\zeta=1}^2 \left[c_\zeta(k; \alpha) u^{(\zeta)}(k) e^{-ikx} + d_\zeta^\dagger(k; \alpha) v^{(\zeta)}(k) e^{ikx} \right],$$

where $u^{(\zeta)}(k)$ and $v^{(\zeta)}(k)$ are basic spinors. The fermion field $\psi(x; \alpha)$ is defined by

$$\psi(x; \alpha) = U(\alpha) \psi(x) U^{-1}(\alpha),$$

where $U(\alpha)$ is

$$\begin{aligned} U(\alpha) &= \exp \left\{ \sum_k \left\{ \theta_c(k; \alpha) [c^\dagger(k) \tilde{c}^\dagger(k) - c(k) \tilde{c}(k)] \right. \right. \\ &\quad \left. \left. + \theta_d(k; \alpha) [d^\dagger(k) \tilde{d}^\dagger(k) - d(k) \tilde{d}(k)] \right\} \right\} \\ &= \prod_k U_c(k; \alpha) U_d(k; \alpha), \end{aligned}$$

with

$$\begin{aligned} U_c(k; \alpha) &= \exp \{ \theta_c(k; \alpha) [c^\dagger(k) \tilde{c}^\dagger(k) - c(k) \tilde{c}(k)] \}, \\ U_d(k; \alpha) &= \exp \{ \theta_d(k; \alpha) [d^\dagger(k) \tilde{d}^\dagger(k) - d(k) \tilde{d}(k)] \}. \end{aligned}$$

The fermion α -operators $c(k; \alpha)$ and $d(k; \alpha)$ are written in terms of non α -operators by

$$\begin{aligned} c(k; \alpha) &= U(\alpha)c(k)U^{-1}(\alpha) = U(k; \alpha)c(k)U^{-1}(k; \alpha) \\ &= u_c(k; \alpha)c(k) - v_c(k; \alpha)\tilde{c}^\dagger(k), \\ d(k; \alpha) &= U(\alpha)d(k)U^{-1}(\alpha) = U(k; \alpha)d(k)U^{-1}(k; \alpha) \\ &= u_d(k; \alpha)d(k) - v_d(k; \alpha)\tilde{d}^\dagger(k). \end{aligned}$$

The parameters $\theta_c(k; \alpha)$ and $\theta_d(k; \alpha)$ are such that $\sin \theta_c(k; \alpha) = v_c(k; \alpha)$, and $\sin \theta_d(k; \alpha) = v_d(k; \alpha)$, resulting in $v_c^2(k; \alpha) + u_c^2(k; \alpha) = 1$ and $v_d^2(k; \alpha) + u_d^2(k; \alpha) = 1$. The inverse formulas for the α -operators are

$$\begin{aligned} c(k) &= u_c(k; \alpha)c(k; \alpha) + v_c(k; \alpha)\tilde{c}^\dagger(k; \alpha), \\ d(k) &= u_d(k; \alpha)d(k; \alpha) + v_d(k; \alpha)\tilde{d}^\dagger(k; \alpha). \end{aligned}$$

Observe that the operators c and d carry a spin index.

These operators satisfy the anti-commutation relations

$$\{c_\zeta(k, \alpha), c_\zeta^\dagger(k', \alpha)\} = \{d_\zeta(k, \alpha), d_\zeta^\dagger(k', \alpha)\} = (2\pi)^3 \frac{k_0}{m} \delta(\mathbf{k} - \mathbf{k}') \delta_{\zeta\zeta'},$$

with all the other anti-commutation relations being zero. In order to be consistent with the Lie algebra, and with the definition of the α -operators, a fermion operator, A , is such that $\tilde{\tilde{A}} = -A$ and a tilde-fermion operator anti-commutes with a non-tilde operator. This is consistent in the following sense. Consider, for instance, $c(k; \alpha) = U(k; \alpha)c(k)U^{-1}(k; \alpha)$ and $\tilde{c}(k; \alpha) = U(k; \alpha)\tilde{c}(k)U^{-1}(k; \alpha)$. In order to map $c(k; \alpha) \rightarrow \tilde{c}(k; \alpha)$ by using the modular conjugation, directly, it leads to $\tilde{\tilde{c}}(k) = -c(k)$. This is important to preserve the canonical structure of $U(k; \alpha)$, regarding in particular the * Lie-algebra. This analysis provides then a precise and simple way to define the modular conjugation for fermions.

Let us define the α -state $|0(\alpha)\rangle = U(\alpha)|0, \tilde{0}\rangle$, where

$$|0, \tilde{0}\rangle = \bigotimes_k |0, \tilde{0}\rangle_k$$

and $|0, \tilde{0}\rangle_k$ is the vacuum for the mode k for particles and anti-particles. This α -state satisfies the condition $\langle 0(\alpha)|0(\alpha)\rangle = 1$. Moreover, we have

$$\begin{aligned} c(k; \alpha)|0(\alpha)\rangle &= \tilde{c}(k; \alpha)|0(\alpha)\rangle = 0, \\ d(k; \alpha)|0(\alpha)\rangle &= \tilde{d}(k; \alpha)|0(\alpha)\rangle = 0. \end{aligned}$$

Then $|0(\alpha)\rangle$ is a vacuum state for the α -operators $c(k; \alpha)$ and $d(k; \alpha)$. Basis vectors are given in the form

$$[c^\dagger(k_1; \alpha)]^{r_1} \cdots [d^\dagger(k_M; \alpha)]^{r_M} [\tilde{c}^\dagger(k_1; \alpha)]^{s_1} \cdots [\tilde{d}^\dagger(k_N; \alpha)]^{s_N} |0(\alpha)\rangle,$$

where $r_i, s_i = 0, 1$. A general α -state can then be defined by a linear combinations of such basis vectors.

Let us consider some particular cases, first, the case of temperature. The topology is Γ_D^1 , and we take $\alpha = (\beta, 0, \dots, 0)$, leading to

$$v_c^2(k^0; \beta) = \frac{1}{e^{\beta(w_k - \mu_c)} + 1},$$

$$v_d^2(k^0; \beta) = \frac{1}{e^{\beta(w_k + \mu_d)} + 1},$$

where μ_c and μ_d are the chemical potential for particles and antiparticles, respectively. For simplicity, we take $\mu_c = \mu_d = 0$, and write $v_F(k^0; \beta) = v_c(k^0; \beta) = v_d(k^0; \beta)$, such that

$$v_F^2(k^0; \beta) = \frac{1}{e^{\beta w_k} + 1} = \sum_{n=1}^{\infty} (-1)^{1+n} e^{-\beta w_k n}.$$

For the case of spatial compactification, we take $\alpha = (0, iL_1, 0, \dots, 0)$. By a kind of Wick rotation, we derive $v_F^2(k^1; L_1)$ from $v_F^2(k^0; \beta)$, resulting in

$$v_F^2(k^1; L_1) = \sum_{n=1}^{\infty} (-1)^{1+n} e^{-iL_1 k^1 n}.$$

For spatial compactification and temperature, we have (see the Appendix)

$$v_F^2(k^0, k^1; \beta, L_1) = v_F^2(k^1; \beta) + v_F^2(k^1; L_1) + 2v_F^2(k^1; \beta)v_F^2(k^1; L_1).$$

The α -Green function is defined by $S_0(x, y; \alpha) = w_A^\alpha(x, y) \equiv \langle 0(\alpha) | A(x, y) | 0(\alpha) \rangle$, where $A(x, y) = T[\psi(x)\bar{\psi}(y)]$. Then we have

$$S_0(x - y; \alpha) = -i \langle 0(\alpha) | T[\psi(x)\bar{\psi}(y)] | 0(\alpha) \rangle. \quad (38)$$

Let us write

$$iS_0(x - y; \alpha) = \theta(x^0 - y^0)S(x - y; \alpha) - \theta(y^0 - x^0)\bar{S}(y - x; \alpha), \quad (39)$$

with $S(x - y; \alpha) = \langle 0(\alpha) | \psi(x)\bar{\psi}(y) | 0(\alpha) \rangle$ and $\bar{S}(x - y; \alpha) = \langle 0(\alpha) | \bar{\psi}(y)\psi(x) | 0(\alpha) \rangle$. Calculating S and \bar{S} we obtain

$$S(x-y; \alpha) = (i\gamma \cdot \partial + m) \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{2\omega_k} \\ \times [e^{-ik(x-y)} - v_F^2(k_\alpha; \alpha)(e^{-ik(x-y)} - e^{ik(x-y)})],$$

For the term $\bar{S}(x-y; \alpha)$, we have

$$\bar{S}(x-y; \alpha) = (i\gamma \cdot \partial + m) \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{2\omega_k} \\ \times [-e^{ik(x-y)} + v_F^2(k_\alpha; \alpha)(e^{-ik(x-y)} + e^{ik(x-y)})].$$

This leads to

$$S_0(x-y; \alpha) = (i\gamma \cdot \partial + m)G_0^F(x-y; \alpha), \quad (40)$$

where

$$G_0^F(x-y; \alpha) = \int \frac{d^Dk}{(2\pi)^D} e^{-ik(x-y)} G_0^F(k; \alpha), \quad (41)$$

and

$$G_0^F(k; \alpha) = G_0(k) + v_F^2(k_\alpha; \alpha)[G_0(k) - G_0^*(k)]. \quad (42)$$

This Green function is similar to the boson Green function, Eq. (35); the difference is the fermion function $v_F^2(k_\alpha; \alpha)$. Again we observe that, due to the GNS construction, the 2×2 -representation of the propagator is not necessary, although it can be introduced. In such a case the Bogoliubov transformation is written in the form of a 2×2 matrix for particles (subindex c) and anti-particles (subindex d) is

$$B_{c,d}(k_\alpha; \alpha) = \begin{pmatrix} u_{c,d}(k_\alpha; \alpha) & v_{c,d}(k_\alpha; \alpha) \\ -v_{c,d}(k_\alpha; \alpha) & u_{c,d}(k_\alpha; \alpha) \end{pmatrix}. \quad (43)$$

4 Generating functional

We now construct the generating functional for interacting fields living in a flat space with topology Γ_D^d .

4.1 Bosons

For a system of free bosons, we consider, up to normalization factors, the following generating functional

$$Z_0 \simeq \int D\phi e^{iS} = \int D\phi \exp[i \int dx \mathcal{L}] = \int D\phi \exp\{-i \int dx [\frac{1}{2}\phi(\square + m^2)\phi - J\phi]\}, \quad (44)$$

where J is a source. Such a functional is written as

$$Z_0 \simeq \exp\left\{\frac{i}{2} \int dxdy [J(x)(\square + m^2 - i\varepsilon)^{-1} J(y)]\right\}, \quad (45)$$

describing the usual generating functional for bosons. However, we would like to introduce the topology Γ_D^d . This is possible by finding a solution of the Klein-Gordon equation

$$(\square + m^2 + i\varepsilon)G_0(x - y; \alpha) = -\delta(x - y). \quad (46)$$

Using this result in Eq. (45), we find the normalized functional

$$Z_0[J; \alpha] = \exp\left\{\frac{i}{2} \int dxdy [J(x)G_0(x - y; \alpha)J(y)]\right\}. \quad (47)$$

Then we have

$$G_0(x - y; \alpha) = i \frac{\delta^2 Z[J; \alpha]}{\delta J(y) \delta J(x)} \Big|_{J=0}.$$

In order to treat interactions, we analyze the usual approach with the α -Green function. The Lagrangian density is

$$\mathcal{L}(x) = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{m^2}{2} \phi^2 + \mathcal{L}_{int},$$

where $\mathcal{L}_{int} = \mathcal{L}_{int}(\phi)$ is the interaction Lagrangian density. The functional $Z[J; \alpha]$ satisfies the equation

$$(\square + m) \frac{\delta Z[J; \alpha]}{i \delta J(x)} + L_{int} \left(\frac{1}{i} \frac{\delta}{\delta J} \right) Z[J; \alpha] = J(x) Z[J; \alpha]$$

with the normalized solution given by

$$Z[J; \alpha] = \frac{\exp \left[i \int dx L_{int} \left(\frac{1}{i} \frac{\delta}{\delta J} \right) \right] Z_0[J; \alpha]}{\exp \left[i \int dx L_{int} \left(\frac{1}{i} \frac{\delta}{\delta J} \right) \right] Z_0[J; \alpha] \Big|_{J=0}}.$$

Observe that the topology does not change the interaction. This is a consequence of the isomorphism and the fact that we are considering a local interaction. Now we turn our attention to construct the α -generator functional for fermions.

4.2 Fermions

The Lagrangian density for a free fermion system with sources is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi + \bar{\psi}\eta + \bar{\eta}\psi.$$

The functional $Z_0 \simeq \int D\psi D\bar{\psi} e^{iS}$ is then reduced to

$$Z_0[\eta, \bar{\eta}; \alpha] = \exp\{-i \int dxdy [\bar{\eta}(x) S_0(x-y; \alpha) \eta(x)]\}, \quad (48)$$

where

$$S_0(x-y; \alpha)^{-1} = i\gamma^\mu \partial_\mu - m.$$

Since $S_0(x-y; \alpha)^{-1} S_0(x-y; \alpha) = \delta(x-y)$, and $G_0(x-y; \alpha)$ satisfies Eq. (46), we find

$$S_0(x-y; \alpha) = (i\gamma \cdot \partial + m) G_0^F(x-y; \alpha).$$

The functional given in Eq. (48) provides the same expression for the propagator, as derived in the canonical formalism, i.e.

$$S_0(x-y; \alpha) = i \frac{\delta^2}{\delta \bar{\eta} \delta \eta} Z_0[\eta, \bar{\eta}; \alpha] |_{\eta=\bar{\eta}=0}.$$

For interacting fields, we obtain

$$Z[\bar{\eta}, \eta; \alpha] = \frac{\exp \left[i \int dx L_{int} \left(\frac{1}{i} \frac{\delta}{\delta \bar{\eta}}; \frac{1}{i} \frac{\delta}{\delta \eta} \right) \right] Z_0[\eta, \bar{\eta}; \alpha]}{\exp \left[i \int dx L_{int} \left(\frac{1}{i} \frac{\delta}{\delta \bar{\eta}}; \frac{1}{i} \frac{\delta}{\delta \eta} \right) \right] Z_0[\eta, \bar{\eta}; \alpha] |_{\bar{\eta}=\eta=0}}.$$

It is important to note that, when $\alpha \rightarrow \infty$ we have to recover the flat space-time field theory, for both bosons and fermions.

4.3 Gauge fields

The Lagrangian density for quantum chromodynamics is given by

$$\begin{aligned} \mathcal{L} = & \bar{\psi}(x) [iD_\mu \gamma^\mu - m] \psi(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ & - \frac{1}{2\sigma} (\partial^\mu A_\mu^r(x))^2 + A_\mu^r(x) t^r J^\mu(x) + \partial^\mu \chi^*(x) D_\mu \chi(x), \end{aligned}$$

where

$$F_{\mu\nu}^r = \partial_\mu A_\nu^r(x) - \partial_\nu A_\mu^r(x) + g c^{rsl} A_\mu^s(x) A_\nu^l(x)$$

and $F_{\mu\nu} = \sum_r F_{\mu\nu}^r t^r$ is the field tensor describing gluons; t^r and c^{rsl} are, respectively, generators and structure constants of the gauge group $SU(3)$; the covariant derivative is given by $D_\mu = \partial_\mu + ig A_\mu = \partial_\mu + ig A_\mu^r(x) t^r$ and $\psi(x)$ is the quark field, including the flavor and color components. The ghost field is given by $\chi(x)$. The quantity $\frac{1}{2\sigma} (\partial^\mu A_\mu^r(x))^2$ is the gauge term, with σ being the gauge-fixing parameter.

The generating functional using the Lagrangian density \mathcal{L} is

$$Z[J, \eta, \bar{\eta}, \xi, \xi^*] = \int DAD\psi D\bar{\psi} D\chi D\chi^* \\ \times \exp \left[i \int d^4x \left(\mathcal{L} + AJ + \bar{\eta}\psi + \bar{\psi}\eta + \xi^*\chi + \chi^*\xi \right) \right],$$

where ξ^* and ξ are Grassmann variables describing sources for ghost fields, and $\bar{\eta}$ and η are the Grassman-variable sources for quarks fields, and J stands for the source of the gluon-field. Notice that we are using non-tilde fields, in such a way that the propagator is a c-number.

We write the Lagrangian density in terms of interacting and noninteracting parts: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with $\mathcal{L}_0 = \mathcal{L}_0^G + \mathcal{L}_0^{FP} + \mathcal{L}_0^Q$, where \mathcal{L}_0^G is the free gauge field contribution including a gauge fixing term, i.e.

$$\mathcal{L}_0^G = -\frac{1}{4}(\partial_\mu A_\nu^r - \partial_\nu A_\mu^r)(\partial^\mu A^{\nu r} - \partial^\nu A^{\mu r}) - \frac{1}{2\sigma}(\partial^\mu A_\mu^r)^2.$$

The term \mathcal{L}_0^{FP} corresponds to the Faddeev-Popov field,

$$\mathcal{L}_0^{FP} = (\partial^\mu \chi_\mu^{r*})(\partial^\mu \chi_\mu^r),$$

and \mathcal{L}_0^Q describes the quark field,

$$\mathcal{L}_0^F = \bar{\psi}(x)[\gamma \cdot i\partial - m]\psi(x).$$

The interaction term is

$$\mathcal{L}_I = -\frac{g}{2}c^{rsl}(\partial_\mu A_\nu^r - \partial_\nu A_\mu^r)A^{s\mu}A^{l\nu} \\ - \frac{g^2}{2}c^{rst}c^{ult}A_\mu^rA_\nu^sA^{u\mu}A^{l\nu} \\ - gc^{rsl}(\partial^\mu \chi^{r*})A_\mu^l\chi^s(y) + g\bar{\psi}t^r\gamma^\mu A_\mu^r\psi.$$

Following steps similar to those in the scalar field case, we write for the gauge field

$$Z_0^{G(rs)}[J] = \exp\left\{\frac{i}{2} \int dx dy [J^\mu(x) D_{0\mu\nu}^{(rs)}(x-y; \alpha) J^\nu(y),\right.$$

where

$$D_0^{(rs)\mu\nu}(x; \alpha) = \int \frac{d^D k}{(2\pi)^D} e^{-ikx} D_0^{(rs)\mu\nu}(k; \alpha)$$

with

$$D_0^{(rs)\mu\nu}(k; \alpha) = \delta^{rs} d^{\mu\nu}(k) G_0(x-y; \alpha),$$

and

$$d^{\mu\nu}(k) = g^{\mu\nu} - (1 - \sigma) \frac{k^\mu k^\nu}{k^2}.$$

For the Fadeev-Popov field we have

$$Z_0^{FP}[\bar{\xi}, \xi] = \exp\left\{\frac{i}{2} \int dxdy [\bar{\xi}(x) G_0(x-y; \alpha) \xi(y)]\right\},$$

where $\bar{\xi}$ and ξ are Grassmann variables. It is important to note that $G_0(x-y; \alpha)$ is the propagator for the scalar field. Then we write the full generating functional for the non-abelian gauge field as

$$Z[J, \bar{\xi}, \xi, \bar{\eta}, \eta] = \frac{\mathcal{E}[\partial_{source}] Z_0[J, \bar{\xi}, \xi, \bar{\eta}, \eta]}{\mathcal{E}[\partial_{source}] Z_0[0]},$$

where

$$\mathcal{E}[\partial_{source}] = \exp\left[i \int dx L_{int} \left(\frac{1}{i} \frac{\delta}{\delta J}, \frac{1}{i} \frac{\delta}{\delta \bar{\xi}}, \frac{1}{i} \frac{\delta}{\delta \xi}, \frac{1}{i} \frac{\delta}{\delta \bar{\eta}}, \frac{1}{i} \frac{\delta}{\delta \eta} \right)\right]$$

and

$$Z_0[0] = Z_0^{G(rs)}[J] Z_0^{FP(rs)}[\bar{\xi}, \xi] Z_0^{F(rs)}[\bar{\eta}, \eta] |_{J=\bar{\xi}=\xi=\bar{\eta}=\eta=0}.$$

As an example, we derive the gluon-quark-quark three point function to order g ,

$$\begin{aligned} G_\mu^a(x_1, x_2, x_3; \alpha) &= -t^a \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \\ &\quad \times \exp i\{-p_1 \cdot x_1 + p_2 \cdot x_2 + (p_1 - p_2) \cdot x_3\} d_{\mu\nu} \\ &\quad \times S_0(p_1; \alpha) \gamma^\nu S_0(p_2; \alpha) D_0(p_1 - p_2; \alpha). \end{aligned}$$

We observe that $G_\mu^a(x_1, x_2, x_3; \alpha)$ has a part independent of the topology, i.e. the flat space contribution $G_\mu^a(x_1, x_2, x_3)$. This is due to the form of the integral representation of the propagators $S_0(p_1; \alpha)$ and $D_0(p_1 - p_2; \alpha)$ and represent a general property of the theory.

5 S -Matrix and reaction rates

In this section we explore the notion of the S -matrix, using the GNS construction as presented earlier. We use the canonical formalism for abelian fields and derive the reaction rate formulas as functions of parameters describing the space-time compactification, particularizing our discussion to the 4-dimensional Minkowski space.

5.1 *S*-Matrix

Consider a field operator $\phi(x)$ such that

$$\begin{aligned}\lim_{t \rightarrow -\infty} \phi(x; \alpha) &= \phi_{in}(x; \alpha), \\ \lim_{t \rightarrow \infty} \phi(x; \alpha) &= \phi_{out}(x; \alpha),\end{aligned}$$

where $\phi_{in}(x; \alpha)$ and $\phi_{out}(x; \alpha)$ stand for the in- and out-fields before and after interaction takes place, respectively. These two fields are assumed to be related by a canonical transformation

$$\phi_{out}(x; \alpha) = S^{-1} \phi_{in}(x; \alpha) S,$$

where S is a unitary operator.

We define the evolution operator, $U(t, t')$, relating the interacting field to the incoming field, *i.e*

$$\phi(x; \alpha) = U^{-1}(t, -\infty) \phi_{in}(x; \alpha) U(t, -\infty), \quad (49)$$

with $U(-\infty, -\infty) = 1$. The operator $\phi(x; \alpha)$ satisfies the Heisenberg equation

$$-i\partial_t \phi(x; \alpha) = [\widehat{H}, \phi(x; \alpha)],$$

where the generator of time translation, \widehat{H} , is written as $\widehat{H} = \widehat{H}_0 + \widehat{H}_I$, with H_0 and H_I being the free-particle and interaction Hamiltonians, respectively. The field $\phi_{in}(x; \alpha)$ satisfies

$$-i\partial_t \phi_{in}(x; \alpha) = [\widehat{H}_0, \phi_{in}(x; \alpha)]. \quad (50)$$

Requiring unitarity of $U(t, t')$, we have

$$\partial_t (U(t, t') U^{-1}(t, t')) = 0.$$

In addition, from Eq. (49) we have

$$\begin{aligned}\partial_t \phi_{in}(x; \alpha) &= \partial_t [U(t, -\infty) \phi(x; \alpha) U^{-1}(t, -\infty)] \\ &= [U(t, -\infty) \partial_t U^{-1}(t, -\infty) + i\widehat{H}, \phi_{in}(x; \alpha)].\end{aligned}$$

Comparing with Eq. (50), we obtain

$$i\partial_t U(t, -\infty) = \widehat{H}_I(t) U(t, -\infty).$$

This equation is written as,

$$U(t, -\infty) = I - i \int_{-\infty}^t dt_1 \widehat{H}_I(t_1) U(t_1, -\infty),$$

that is solved by iteration, resulting in

$$\begin{aligned}
U(t, -\infty) &= I - i \int_{-\infty}^t dt_1 \widehat{H}_I(t_1) + (-i)^2 \int_{-\infty}^t \int_{-\infty}^{t_1} dt_1 dt_2 \widehat{H}_I(t_1) \widehat{H}_I(t_2) + \dots \\
&\quad + (-i)^n \int_{-\infty}^t \dots \int_{-\infty}^{t_{n-1}} dt_1 \dots dt_n \widehat{H}_I(t_1) \dots \widehat{H}_I(t_n) + \dots \\
&= T \exp \left[-i \int_{-\infty}^t dt' \widehat{H}_I(t') \right],
\end{aligned}$$

where T is the time-ordering operator.

The S -matrix is defined by $S = \lim_{t \rightarrow \infty} U(t, -\infty)$, such that $S = \sum_{n=0}^{\infty} S^{(n)}$, where

$$S^{(n)} = \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dt_1 \dots dt_n T \left[\widehat{H}_I(t_1) \dots \widehat{H}_I(t_n) \right].$$

Then we have

$$S = T \exp \left[-i \int_{-\infty}^{\infty} dt' \widehat{H}_I(t') \right].$$

The transition operator, \mathcal{T} , is defined by $\mathcal{T} = S - I$. Observe that $\widehat{H}(\alpha) \equiv \widehat{H}$, and in the definition of the S -matrix there is no need to introduce a tilde S -matrix, as is the case of TFD [7]. Here this is a consequence of the GNS construction.

5.2 Reaction rates

Consider the scattering process

$$p_1 + p_2 + \dots + p_r \rightarrow p'_1 + p'_2 + \dots + p'_r,$$

where p_i and p'_i are momenta of the particles in the initial and final state, respectively. The amplitude for this process is obtained by the usual Feynman rules as

$$\langle f | S | i \rangle = \sum_{n=0}^{\infty} \langle f | S^{(n)} | i \rangle,$$

where $|i\rangle = a_{p_1}^\dagger a_{p_2}^\dagger \dots a_{p_r}^\dagger |0\rangle$ and $|f\rangle = a_{p'_1}^\dagger a_{p'_2}^\dagger \dots a_{p'_r}^\dagger |0\rangle$ with $|0\rangle$ being the vacuum state, such that $a_p |0\rangle = 0$. For the topology Γ_D^d , a similar procedure may be used by just replacing $|i\rangle$ and $|f\rangle$ states for $|i; \alpha\rangle$ and $|f; \alpha\rangle$. The amplitude for the process is then given as

$$\langle f; \alpha | \hat{S} | i; \alpha \rangle = \sum_{n=0}^{\infty} \langle f; \alpha | \hat{S}^n | i; \alpha \rangle,$$

where

$$\begin{aligned} |i; \alpha\rangle &= a_{p_1}^\dagger(\alpha) a_{p_2}^\dagger(\alpha) \dots a_{p_r}^\dagger(\alpha) |0(\alpha)\rangle, \\ |f; \alpha\rangle &= a_{p'_1}^\dagger(\alpha) a_{p'_2}^\dagger(\alpha) \dots a_{p'_r}^\dagger(\alpha) |0(\alpha)\rangle, \end{aligned}$$

The vacuum state in the topology Γ_D^d is given by $|0(\alpha)\rangle$. As emphasized earlier, the phase-space factors are not changed by the topology. The meaning of these states is described in Section IV.

The differential cross-section for the particular process

$$p_1 + p_2 \rightarrow p'_1 + p'_2 + \dots + p'_r$$

is given by

$$\begin{aligned} d\sigma &= (2\pi)^4 \delta^4(p'_1 + p'_2 + p'_3 + \dots + p'_r - p_1 - p_2) \\ &\times \frac{1}{4E_1 E_2 v_{rel}} \prod_l (2m_l) \prod_{j=1}^r \frac{d^3 p'_j}{(2\pi)^3 2E'_j} |M_{fi}(\alpha)|^2, \end{aligned} \quad (51)$$

where $E'_j = \sqrt{m_j'^2 + \mathbf{p}_j'^2}$ and v_{rel} is the relative velocity of the two initial particles with 3-momenta \mathbf{p}_1 and \mathbf{p}_2 . The factor $2m_j$ appears for each lepton in the initial and final state. Here E_1 and E_2 are the energies of the two particles with momenta \mathbf{p}_1 and \mathbf{p}_2 , respectively. The amplitude M_{fi} is related to the S -matrix element by

$$\langle f; \alpha | S | i; \alpha \rangle = i (2\pi)^4 M_{fi}(\alpha) \prod_i \left(\frac{1}{2V E_i} \right)^{\frac{1}{2}} \prod_f \left[\frac{1}{2V E_f} \right]^{\frac{1}{2}} \delta^4(p_f - p_i). \quad (52)$$

Here p_f and p_i are the total 4-momenta in the final and initial state, respectively. The product extends over all the external fermions and bosons, with E_i and E_f being the energy of particles in the initial and final states, respectively and V is the volume.

5.3 Decay of particles

Consider the decay of the boson field σ into π , with an interaction Lagrangian density given by

$$\mathcal{L}_I = \lambda \sigma(x) \pi(x) \pi(x). \quad (53)$$

The initial and final states in Γ_D^d are, respectively,

$$|i; \alpha\rangle = a_k^\dagger(\alpha) |0(\alpha)\rangle,$$

and

$$|f; \alpha\rangle = b_{k_1}^\dagger(\alpha) b_{k_2}^\dagger(\alpha) |0(\alpha)\rangle,$$

where $a_k^\dagger(\alpha)$ and $b_k^\dagger(\alpha)$ are creation operators in the topology Γ_D^d for the σ - and π - particles, respectively, with momenta k . At the tree level, the transition matrix element is

$$\begin{aligned} \langle f; \alpha | \hat{S} | i; \alpha \rangle &= i\lambda \int dx \langle 0(\alpha) | b_{k_2}(\alpha) b_{k_1}(\alpha) \\ &\times [\sigma(x)\pi(x)\pi(x) - \tilde{\sigma}(x)\tilde{\pi}(x)\tilde{\pi}(x)] a_k^\dagger(\alpha) | 0(\alpha) \rangle. \end{aligned}$$

Using the expansion of the boson fields, $\sigma(x)$ and $\pi(x)$, in momentum space, the Bogoliubov transformation and the commutation relations, the two terms of the matrix element are calculated. For instance we have

$$\langle 0(\alpha) | \sigma(x) a_k^\dagger(\alpha) | 0(\alpha) \rangle = e^{-ikx} \cosh \theta(k; \alpha),$$

Combining these factors, the amplitude for the process is

$$\begin{aligned} M_{fi}(\beta) &= \lambda [\cosh(k; \alpha) \cosh \theta(k_1; \alpha) \cosh \theta(k_2; \alpha) \\ &\quad - \sinh \theta(k; \alpha) \sinh \theta(k_1; \alpha) \sinh \theta(k_2; \alpha)]. \end{aligned}$$

Note that the indices “1” and “2” in k_1 and k_2 are refereing here to particles.

The decay rate for the σ -meson is given as

$$\begin{aligned} \Gamma(w, \alpha) &= \frac{1}{2w} \int \frac{d^3 k_1 d^3 k_2 (2\pi)^4 \delta^4(k - k_1 - k_2)}{(2w_1)(2w_2)(2\pi)^3(2\pi)^3} |M_{fi}(\alpha)|^2 \\ &= \frac{\lambda^2}{32w\pi^2} \int \frac{d^3 k_1}{w_1} \frac{d^3 k_2}{w_2} \delta^4(k - k_1 - k_2) W(w; w_1, w_2; \alpha), \end{aligned} \quad (54)$$

where

$$\begin{aligned} W(w; w_1, w_2; \alpha) &= |\cosh \theta(k; \alpha) \cosh \theta(k_1; \alpha) \cosh \theta(k_2; \alpha) \\ &\quad - \sinh \theta(k; \alpha) \sinh \theta(k_1; \alpha) \sinh \theta(k_2; \alpha)|^2, \end{aligned}$$

with

$$w_i = \sqrt{\kappa_i^2 + m^2}, w = \sqrt{\mathbf{k}^2 + M^2}.$$

Using $\sinh^2 \theta(k; \alpha) = v^2(k; \alpha) \equiv n(k; \alpha)$ and $\cosh^2 \theta(k; \alpha) = u^2(k; \alpha) \equiv 1 + n(k; \alpha)$, we have

$$\begin{aligned}
W(w; w_1, w_2; \alpha) = & \left| \sqrt{1 + n(k; \alpha)} \right|^2 \left| \sqrt{1 + n(k_1; \alpha)} \right|^2 \left| \sqrt{1 + n(k_2; \alpha)} \right|^2 \\
& + \left| \sqrt{n(k; \alpha)} \right|^2 \left| \sqrt{n(k_1; \alpha)} \right|^2 \left| \sqrt{n(k_2; \alpha)} \right|^2 \\
& - 2 \Re \{ [1 + n(k; \alpha)][1 + n(k_1; \alpha)][1 + n(k_2; \alpha)] \\
& \times n(k; \alpha)n(k_1; \alpha)n(k_2; \alpha) \}^{1/2}.
\end{aligned} \tag{55}$$

Considering the rest frame of the decaying particle: $w = M$, $\mathbf{k} = 0$, $w_i = \sqrt{\mathbf{k}_i^2 + m^2} = \sqrt{\mathbf{q}^2 + m^2} = w_q$, and the case of temperature only, i.e. $\alpha = (\beta, 0, 0, 0)$, we recover the result derived in Ref. [56].

One aspect to be emphasized is the notion of quasi-particles. The energy spectrum of the particles taking place in the reaction has changed as a consequence of the compactification. This new spectrum corresponds to the energy of the quasi-particles. The broken symmetry here is due to a topological specification in the Minkowski space-time. This interpretation is valid also for the thermal effects, considered from a topological point of view.

6 Concluding remarks

In this paper we have developed a theory for quantum fields defined on a D -dimensional space-time having a topology $\Gamma_D^d = (\mathbb{S}^1)^d \times \mathbb{M}^{D-d}$, with $1 \leq d \leq D$. This describes simultaneously spatial constraints and thermal effects. We use the modular construction of c^* -algebra as a key tool. With the modular group, we study a $*$ -representation of Lie algebras, specifically analyzing the Poincaré group. The propagator for bosons and fermions is found to be a generalization of the Fourier integral representation [4,5] of the imaginary-time propagator. Some results deserve to be emphasized.

- (i) Considering $d = 1$, with the compactification parameter being $\beta = 1/T$, we show that it is possible to develop a consistent real-time quantum field theory at finite temperature T , where the propagator is given in the Fourier integral representation. Therefore, there is no need to use a 2×2 matrix structure for the propagator and for the generating functional, as is the case in TFD and in the Schwinger-Keldysh approach. This is a simplification in the formalism, that is explored here for the general case of the topology Γ_D^d .
- (ii) Using the modular representation, we study the Poincaré group. The modular conjugation is defined in order to respect the Lie algebra structure. This procedure provides a precise way to define the modular conjugation for fermions [51].
- (iii) The extension of the formalism for abelian and non-abelian gauge-fields is developed using functional methods.

- (iv) The S-matrix is introduced, and as an application we calculate the space-compactification effect in a decay process.
- (v) The compactification is described as a process of condensation in the vacuum state, $|0(\alpha)\rangle$. The parameter α describes the topological effects, which modify the energy spectrum, giving rise to the notion of quasi-particles.
- (vi) The Fourier integral representation of the propagator is separated into a divergent part and a finite contribution from the topological effects. This feature has proved to be useful in the study of numerous processes [7,12].

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A Fourier integral representation of the Green function

Consider a scalar field at finite temperature, such that the Green function satisfies the Klein-Gordon equation written in a D -dimensional Minkowski space,

$$(\square + m^2)G(x - y; \beta) = -\delta(x - y). \quad (\text{A.1})$$

We take the Fourier integral representation of the Green function as derived in Refs. [4] and [5], written for \mathbb{M}^D ,

$$G_0(x - y; \beta) = \int \frac{d^D p}{(2\pi)^D} G_0(p; \beta), \quad (\text{A.2})$$

where

$$G_0(p; \beta) = G_0(p) + f_\beta(p^0)[G_0(p) - G_0^*(p)],$$

where

$$f_\beta(p^0) = \frac{1}{e^{\beta p^0} - 1} = \sum_{l_0=1}^{\infty} e^{-\beta p^0 l_0}$$

and

$$G_0(k) = \frac{-1}{p^2 - m^2 + i\varepsilon}.$$

The thermal theory corresponds to a topology $\mathbb{S}^1 \times \mathbb{M}^{D-1}$. Now we would like to generalize the theory to the topology $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{M}^{D-2}$, *i.e.*, compactification of time and one spatial dimension. We choose that the x^1 direction is compactified with a period L_1 . Observe that the topology does not change the local properties of the system. This implies that locally the Minkowski space, as well as a differential equation defined by an isometry, such as the Klein-Gordon equation, are the same. However, the topology imposes modifications on boundary conditions to be fulfilled by the field and the respective Green function.

The new Green function satisfies the periodic boundary condition,

$$G(x^0, x^1, \mathbf{x}; \beta) \equiv G(x^0, x^1 + L_1, \mathbf{x}; \beta) = G(x + L_1 \hat{n}_1; \beta), \quad (\text{A.3})$$

where $\hat{n}_1 = (n_1^\mu) = (0, 1, 0, \dots, 0)$ and $\mathbf{x} = (x^2, \dots, x^{D-1})$. A solution of Eq. (A.1), satisfying this condition, is obtained by using the Fourier expansion

$$G(x - y; L_1; \beta) = \frac{1}{L_1} \sum_{n=-\infty}^{\infty} \int \frac{dp_0 d^{D-2} \mathbf{p}}{(2\pi)^{D-1}} e^{-ip_n(x-y)} G_0(p_n; L_1; \beta), \quad (\text{A.4})$$

where

$$p_n = (p_0, p_{1n}, \mathbf{p}), \quad p_{1n} = \frac{2\pi n}{L_1}, \quad \mathbf{p} = (p_2, \dots, p_{D-1}),$$

$$G_0(p_n; L_1; \beta) = G_0(p_n; L_1) + f_\beta(p^0)[G_0(p_n; L_1) - G_0^*(p_n; L_1)] \quad (\text{A.5})$$

and

$$G_0(p_n; L_1) = \frac{-1}{p_n^2 - m^2 + i\varepsilon}. \quad (\text{A.6})$$

Inversely, we have,

$$G(p_n; L_1; \beta) = \int_0^{L_1} dx^1 \int dx^0 d^{D-2} \mathbf{x} e^{ip_n x} G(x; L_1; \beta). \quad (\text{A.7})$$

To obtain the Fourier integral representation of $G(x - y; L_1; \beta)$, we first perform a sort of Wick rotation such that $L_1 \rightarrow -iL'_1$; this allows us to proceed in a similar fashion as in the case of temperature. Then, we write $G(x - y; L'_1; \beta)$ as

$$G(x - y; L'_1; \beta) = \theta(x^1 - y^1) G^>(x - y; L'_1; \beta) + \theta(y^1 - x^1) G^<(x - y; L'_1; \beta) \quad (\text{A.8})$$

and, from Eq. (A.3), we have

$$G^<(x; L'_1; \beta) |_{x^1=0} = G^>(x; L'_1; \beta) |_{x^1=-iL'_1}. \quad (\text{A.9})$$

Then, Eq. (A.7) reads

$$G(p'_n; L'_1; \beta) = \int_0^{-iL'_1} dx^1 \int dx^0 d^{D-2} \mathbf{x} e^{ip'_n x} G^>(x; L'_1; \beta), \quad (\text{A.10})$$

where $p'_{1n} = 2\pi n/(-iL'_1)$. The Fourier integral transform of $G(x - y; L'_1; \beta)$, denoted by $\overline{G}(p; L'_1; \beta)$, is

$$\overline{G}(p; L'_1; \beta) = \overline{G}^{(1)}(p; L'_1; \beta) + \overline{G}^{(2)}(p; L'_1; \beta), \quad (\text{A.11})$$

where

$$\overline{G}^{(1)}(p; L'_1; \beta) = \int d^D x e^{ipx} \theta(x^1) G^>(x; L'_1; \beta), \quad (\text{A.12})$$

$$\overline{G}^{(2)}(p; L'_1; \beta) = \int d^D x e^{ipx} \theta(-x^1) G^<(x; L'_1; \beta). \quad (\text{A.13})$$

Writing

$$G^>(x; L'_1; \beta) = \int \frac{d^D p}{(2\pi)^D} e^{-ipx} \overline{G}^>(p; L'_1; \beta) \quad (\text{A.14})$$

and using the integral representation of the step function,

$$\int dk^1 \frac{e^{-ik^1 x^1}}{k^1 + p^1 + i\varepsilon} = (-2\pi i) e^{ip^1 x^1} \theta(x^1),$$

in Eq. (A.12), we have

$$\overline{G}^{(1)}(p; L'_1; \beta) = i \int \frac{dk^1}{2\pi} \frac{\overline{G}^>(p_0, k_1, \mathbf{p}; L'_1; \beta)}{k^1 - p^1 + i\varepsilon}. \quad (\text{A.15})$$

With

$$G^<(x; L'_1; \beta) = \int \frac{d^D p}{(2\pi)^D} e^{-ipx} \overline{G}^<(p; L'_1; \beta),$$

and using the integral representation of the step function,

$$\int dk^1 \frac{e^{-ik^1 x^1}}{k^1 + p^1 - i\varepsilon} = 2\pi i e^{ip^1 x^1} \theta(-x^1),$$

in Eq. (A.13), we obtain

$$\overline{G}^{(2)}(p; L'_1; \beta) = -i \int \frac{dk^1}{2\pi} \frac{\overline{G}^<(p_0, k_1, \mathbf{p}; L'_1; \beta)}{k^1 - p^1 - i\varepsilon}. \quad (\text{A.16})$$

Substituting Eqs. (A.15) and (A.16) in Eq. (A.11), we get

$$\overline{G}(p; L'_1; \beta) = i \int \frac{dk^1}{2\pi} \left[\frac{\overline{G}^>(p_0, k_1, \mathbf{p}; L'_1; \beta)}{k^1 - p^1 + i\varepsilon} - \frac{\overline{G}^<(p_0, k_1, \mathbf{p}; L'_1; \beta)}{k^1 - p^1 - i\varepsilon} \right] \quad (\text{A.17})$$

From the periodicity of the Green function, we have

$$\overline{G}^<(p; L'_1; \beta) = e^{L'_1 p^1} \overline{G}^>(p; L'_1; \beta). \quad (\text{A.18})$$

Defining

$$f_{L'_1}(p^1) = \frac{1}{e^{L'_1 p^1} - 1} = \sum_{l_1=1}^{\infty} e^{-L'_1 p^1 l_1},$$

we write

$$\overline{G}^>(p; L'_1; \beta) = f_{L'_1}(p^1) A(p; L'_1; \beta) \quad (\text{A.19})$$

$$\overline{G}^<(p; L'_1; \beta) = [f_{L'_1}(p^1) + 1] A(p; L'_1; \beta). \quad (\text{A.20})$$

Then we have

$$A(p; L'_1; \beta) = \overline{G}^<(p; L'_1; \beta) - \overline{G}^>(p; L'_1; \beta).$$

With these results, Eq. (A.17) reads

$$\begin{aligned} \overline{G}(p; L'_1, \beta) = i \int \frac{dk^1}{2\pi} \left[\frac{f_{L'_1}(k^1) A(p_0, k_1, \mathbf{p}; L'_1; \beta)}{k^1 - p^1 + i\varepsilon} \right. \\ \left. - \frac{[f_{L'_1}(k^1) + 1] A(p_0, k_1, \mathbf{p}; L'_1; \beta)}{k^1 - p^1 - i\varepsilon} \right]. \end{aligned} \quad (\text{A.21})$$

We do not have as yet an explicit expression for $A(p; L'_1; \beta)$. To determine this function, we use the fact that we know $G(p'_n; L'_1; \beta)$, as given in Eq. (A.10). Using Eq. (A.14), we have

$$G(p'_n; L'_1; \beta) = \int_0^{-iL'_1} dx^1 \int dx^0 d^{D-2} \mathbf{x} e^{ip'_n x} \int \frac{d^D k}{(2\pi)^D} e^{-ikx} \overline{G}^>(k; L'_1; \beta).$$

From Eq. (A.19) and the integral representation

$$\int_0^{-iL'_1} dx^1 e^{-i(p'_n - k^1)x^1} = \frac{1}{f_{L'_1}(k^1)} \frac{i}{p'_n - k^1},$$

we obtain

$$G(p'_n; L'_1; \beta) = i \int \frac{dk^1}{2\pi} \frac{A(p_0, k_1, \mathbf{p}; L'_1; \beta)}{p'_n - k^1},$$

where $A(p; L'_1; \beta)$ is the generalization of the spectral function associated with the momentum p^1 .

We consider the analytic continuation of $G(p'_n; L'_1; \beta)$ to take p'_n to be a continuum variable, p^1 . The only possible analytical continuation of $G(p'_n; L'_1; \beta)$ without essential singularity at $p \rightarrow \infty$ is the function

$$\mathcal{G}_0(p; L'_1; \beta) = i \int \frac{dk^1}{2\pi} \frac{A(p_0, k_1, \mathbf{p}; L'_1; \beta)}{p^1 - k^1},$$

where, by definition,

$$\mathcal{G}_0(p; L'_1; \beta) = G_0(p; L'_1; \beta). \quad (\text{A.22})$$

Using this result, we calculate $A(p)$ by showing that

$$\begin{aligned} \mathcal{G}(p; \varepsilon) &= \mathcal{G}_0(p_0, p^1 + i\varepsilon, \mathbf{p}; L'_1; \beta) - \mathcal{G}_0(p_0, p^1 - i\varepsilon, \mathbf{p}; L'_1; \beta) \\ &= i \int \frac{dk^1}{2\pi} A(p_0, k_1, \mathbf{p}; L'_1; \beta) \left[\frac{1}{p^1 - k^1 + i\varepsilon} - \frac{1}{p^1 - k^1 - i\varepsilon} \right] \\ &= i \int \frac{dk^1}{2\pi} A(p_0, k_1, \mathbf{p}; L'_1; \beta) (-2\pi i) \delta(p^1 - k^1). \end{aligned}$$

This leads to

$$A(p; L'_1; \beta) = \mathcal{G}_0(p_0, p^1 + i\varepsilon, \mathbf{p}; L'_1; \beta) - \mathcal{G}_0(p_0, p^1 - i\varepsilon, \mathbf{p}; L'_1; \beta)$$

which describes a discontinuity of $\mathcal{G}_0(p; L_1; \beta)$ across the real axis p^1 .

Using the identity

$$\delta(x^2 - y^2) = \frac{1}{2|y|} [\delta(x + y) + \delta(x - y)],$$

the Fourier integral representation of $G(x - y; L_1'; \beta)$ can be written, after some calculations and transforming back $L_1' \rightarrow iL_1$, as

$$G(x - y; L_1; \beta) = \int \frac{d^D p}{(2\pi)^D} e^{-ip(x-y)} \{G_0(p) + f_{L_1\beta}(p^0, p^1)[G_0(p) - G_0^*(p)]\}, \quad (\text{A.23})$$

where

$$f_{L_1\beta}(p^0, p^1) = f_\beta(p^0) + f_{L_1}(p^1) + 2f_\beta(p^0)f_{L_1}(p^1), \quad (\text{A.24})$$

with

$$f_{L_1}(p^1) = \sum_{l_1=1}^{\infty} e^{-iL_1 p^1 l_1}.$$

In this representation, one important result is that the content of the flat space is given in a separated term involving only $G_0(p)$, while the topological effect of Γ_D^2 is present in the term with $f_{L_1\beta}(p^0, p^1)$, describing compactification of space and time. In addition we obtain: for $L \rightarrow \infty$, $f_{L_1\beta}(p^0, p^1) \rightarrow f_\beta(p^0)$, and for $\beta \rightarrow \infty$, $f_{L_1\beta}(p^0, p^1) \rightarrow f_{L_1}(p^1)$, a consistent result.

For fermions, due to the nature of the statistics, one obtains the same expression as in Eq. (A.24) but with

$$f_\beta(p^0) = \sum_{l_0=1}^{\infty} (-1)^{1+l_0} e^{-\beta p^0 l_0}, \quad f_{L_1}(p^1) = \sum_{l_1=1}^{\infty} (-1)^{1+l_1} e^{-iL_1 p^1 l_1}.$$

The same procedure can be repeated for compactification of several dimensions, corresponding to the topology Γ_D^d , for both boson and fermion fields. For $d (\leq D)$ compactified dimensions, x^0, x^1, \dots, x^{d-1} , the result is the propagator shown in Eqs. (9,10), for bosons, and in Eqs. (40,41,42), for fermions, with $f_\alpha(p^\alpha)$ given by

$$\begin{aligned} f_\alpha(p^\alpha) &= \sum_{s=1}^d \sum_{\{\sigma_s\}} 2^{s-1} \left(\prod_{j=1}^s f_{\alpha_{\sigma_j}}(p^{\sigma_j}) \right) \\ &= \sum_{s=1}^d \sum_{\{\sigma_s\}} 2^{s-1} \sum_{l_{\sigma_1}, \dots, l_{\sigma_s}=1}^{\infty} (-\eta)^{s+\sum_{r=1}^s l_{\sigma_r}} \exp\left\{-\sum_{j=1}^s \alpha_{\sigma_j} l_{\sigma_j} p^{\sigma_j}\right\}, \quad (\text{A.25}) \end{aligned}$$

where $\eta = 1$ (-1) for fermions (bosons) and $\{\sigma_s\}$ denotes the set of all combinations with s elements, $\{\sigma_1, \sigma_2, \dots, \sigma_s\}$, of the first d natural numbers

$\{0, 1, 2, \dots, d-1\}$, that is all subsets containing s elements; in order to obtain the physical condition of finite temperature and spatial confinement, α_0 has to be taken as a positive real number, $\beta = T^{-1}$, while α_n , for $n = 1, 2, \dots, d-1$, must be pure imaginary of the form iL_n .

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